Inequality

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Let $x,y,z > 0$, prove that

$$\sum \frac{x(y+z)}{x+yz} \le 2\sum \frac{x^2}{x+yz}.$$

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Multiplying the inequality by $(x+yz)(y+zx)(z+xy)$ we obtain

$\sum x(y+z)(y+zx)(z+xy) \le 2\sum x^2(y+zx)(z+xy) \iff$

$\sum(x^3y^2z + x^3yz^2) + \sum(x^2y^3 + x^2z^3) + \sum(x^2y^2z + x^2yz^2) + \sum(xy^2z + xyz^2) \le$

$\qquad 2\sum x^4yz + 2\sum(x^3y^2 + x^3z^2) + 2\sum x^2yz \iff$

$\sum(x^3y^2z + x^3yz^2) + \sum(x^2y^2z + x^2yz^2) \le 2\sum x^4yz + \sum(x^3y^2 + x^3z^2).$

Since $x^3 + y^3 \ge x^2y + xy^2 \iff (x+y)(x-y)^2 \ge 0$ we have

$2\sum x^4yz = xyz\sum(x^3 + y^3) \ge xyz\sum(x^2y + xy^2) = xyz\sum x^2(y+z) = \sum x^3yz(y+z).$

Thus,remains to prove inequality

(**1**) $\sum(x^2y^2z + x^2yz^2) \le \sum(x^3y^2 + x^3z^2).$

Since $\sum(x^2y^2z + x^2yz^2) = 2xyz\sum xy$ and $\sum(x^3y^2 + x^2y^3) = \sum x^2y^2 \cdot \sum x - xyz\sum xy$

then (**1**)$\iff 3xyz(xy+yz+zx) \le (x^2y^2 + y^2z^2 + z^2x^2)(x+y+z).$

Assume $x+y+z=1$ (due to homogeneity of inequality (**1**)) and let

$p := xy+yz+zx, q := xyz.$

Then $x^2y^2 + y^2z^2 + z^2x^2 = p^2 - 2q$ and (**1**) becomes $3pq \le p^2 - 2q \iff q(3p+2) \le p^2.$

Since $q = xyz(x+y+z) \le \dfrac{(xy+yz+zx)^2}{3} = \dfrac{p^2}{3}$ and $p = xy+yz+zx \le \dfrac{(x+y+z)^2}{3} = \dfrac{1}{3}$

then $p^2 - q(3p+2) \ge p^2 - \dfrac{p^2}{3}(3p+2) = \dfrac{p^2(1-3p)}{3} \ge 0.$